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TWO NONISOMORPHIC K-AUTOMORPHISMS ALL OF WHOSE POWERS BEYOND ONE ARE ISOMORPHIC

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ABSTRACT

As in the earlier example of two nonisomorphic K-automorphisms with isomorphic square by building an appropriate skew product of a K-automorphism that is not Bernoulli with an algebraic fiber, we get all powers beyond one of the two nonisomorphic transformations to be isomorphic. Furthermore, all the isomorphism maps are finite codings of the constructed partition names.

I. Introduction

We have seen earlier [4] how to construct two nonisomorphic K-automorphisms with isomorphic squares. Using the same sort of skew product structure we can strengthen the construction to make all powers beyond one of the transformations isomorphic. Furthermore, with respect to the partitions which we use to define the transformations, all the isomorphism maps are finite codes, hence are continuous maps on the corresponding sequence spaces.

As in both earlier examples [4], [5] the idea at the core of the skew products comes from a fairly simple example. Consider now the following. Let $p_1, p_2 \cdots p_n$ be the first n primes. Let G^n be the group of all rotations of the circle by amounts $(m_1/p_1 + m_2/p_2 + \cdots + m_n/p_n)$, $0 \leqq m_i < p_i$. It is easy to verify that such rotations have a unique such representation. In G^n select two subsets,

$$G_1^n = \left\{\left(\frac{1}{p_{i_1}} + \frac{1}{p_{i_2}} + \cdots + \frac{1}{p_{i_k}}\right),\ k \text{ odd}\right\} \text{ and}$$

$$G_2^n = \left\{\left(\frac{1}{p_{i_1}} + \frac{1}{p_{i_2}} + \cdots + \frac{1}{p_{i_k}}\right),\ k \text{ even}\right\}.$$

Now G_1^n and G_2^n each contain 2^{n-1} elements, and no element in G_1^n has the same order as an element in G_2^n.

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Let $\Omega_1 = G_1^n \times [0,1)$ and $\Omega_2 = G_2^n \times [0,1)$, and define T_1 on Ω_1 as $T_1(g_1,\theta) = (g_1, g_1+\theta)$ and T_2 on Ω_2 as $T_2(g_2,\theta) = (g_2, g_2+\theta)$. It is clear T_1 and T_2 are nonisomorphic. But for any p_i, $i \leqq n$, $T_1^{p_i}$ and $T_2^{p_i}$ are isomorphic as follows. Let

$$\phi_{p_i}(g_1,\theta) = \begin{cases} \left(g_1 + \frac{1}{p_i}, \ \right) & \text{if } p_i \text{ is not in the sum giving } g_1, \\ \left(g_1 - \frac{1}{p_i}, \ \right) & \text{if } p_i \text{ is in this sum.} \end{cases}$$

We will write this as $g_1 \pm 1/p_i$ to save space. Now

$$\phi_{p_i} \circ T^{p_i}(g_1,\theta) = \phi_{p_i}(g_1, \theta + p_i g_1) = \left(g_i \pm \frac{1}{p_i}, \theta + p_i\left(g_1 \pm \frac{1}{p_i}\right)\right)$$

$$= T_2^{p_i}\left(g_1 \pm \frac{1}{p_i}, \theta\right) = T_2^{p_i} \circ \phi_{p_i}(g_1,\theta).$$

This is the natural extension to higher powers of the trivial example in [4] of isomorphism on squares.

This, of course, only gives isomorphisms on a finite collection of prime powers. For the example we want, we will need it for all primes. To do this we will introduce the primes one at a time on ever longer blocks in the construction, i.e. the finite code giving the isomorphisms for the p_i powers will be on ever longer blocks as i increases. This is necessary as there are only finitely many codes of a bounded length, and if any two of our isomorphisms were identical, the two transformations would be isomorphic.

A further difficulty in the above example is its nonergodicity. If we use such a space as the fiber of a skew product, we will need maps, Ω_1 to Ω_1 and Ω_2 to Ω_2, which preserve the isomorphisms but allow us to make the transformations ergodic. In fact we will make them K-automorphisms.

II. Construction of T_1 and T_2

We will first build a transformation to act as the base of the two skew products. This particular construction is essentially identical to J. Clark's K-automorphism with no roots [1]. Then we will see how to make the two skew products to introduce the isomorphism structure of the above example.

A. *The base map W*

As the previous two examples have given a fairly complete exposition of cutting and stacking, I will be precise here only on those parts which bear directly on the facts we wish to prove.

As usual, a point's $W, (E, F, S_0, S)$ name will be built up in stages, at the $(n+1)$st stage we will give the rule for stringing together n-blocks to form $(n+1)$-blocks.

As was indicated earlier, the necessary structure for the isomorphisms of the prime powers of the transformations will be introduced in stages, p_ith powers on N_i-blocks. Thus there is a need for the nature of the spacing between blocks to change at these points. The values of the N_i, and the spacers for n-blocks, $N_i \leqq n < N_{i+1}$ are as follows. A 0-block is 6 F's followed by 3×2^{100} S_0's, followed by 6 E's and $N_0 = 0$.

If $N_i \leqq n < N_{i+1}$, we form n-blocks out of $(n-1)$-blocks by choosing independently an integer f, $0 < f \times p_i \cdots p_{i+2} < f(n) = n(p_1 \cdots p_{i+2})$, and a sequence of $(2^{i-1})(p_1 \times \cdots \times p_{i+2})^2 \times 2^n$ $(n-1)$-blocks. For this choice, the n-block is $f \times p_1 \cdots p_{i+2}$ F's, the sequence of $(n-1)$-blocks with spacers in between of sizes $S(n)-1$, $2(S(n))-1, \cdots$, $((p_1 \times \cdots \times p_{i+2})^2(2^{i-1})2^n - 1)S(n) - 1$, where $S(n) = (p_1 \times \cdots \times p_{i+2})100n^3$, followed by $f(n) - (f)p_1 \cdots p_{i+2}$ E's at the end. All such combinations of F segments and $(n-1)$-block sequences are possible and each is equally likely. Notice that the F and E segments have lengths $= 0 \bmod p_j$, $j \leqq i+2$, and all spacers have length $= -1 \bmod p_j$, $j \leqq i+2$. Let $h(n)$ be the height of an n-block. We also have $h(n) = 0 \bmod p_j$ if $N_i \leqq n < N_{i+1}$ and $j \leqq i+2$.

We can now state inductively how to choose N_i. Let

$$N_{i+1} = \min\{n \mid (p_1 \times \cdots \times p_{i+2})n > 2^i h(N_i + 1), \text{ and } p_1 \times \cdots \times p_{i+3} < \sqrt{n}\} + 1. \tag{2.1}$$

That is, the isomorphism on the $(i+1)$st prime power is introduced when the randomizing F's at the beginning of an n-block are much larger than the size of an $(N_i + 1)$-block, and the product of the first $i+3$ primes is small compared to all the sizes involved. This now defines W. As is usual, one can show

$$\sum_{i=0}^{n} f(i) < S(n) - 1 \quad \text{and} \quad 2^{10n}(S(n)-1) < h(n-1). \tag{2.2}$$

Now for the skew products.

B. *Making the skew products T_1 and T_2*

Let Ω be the space on which W is defined. Let $\bar{\Omega} = \Omega \times [0,1)_1 \times [0,1)_2$ using

Lebesgue measure on the two circles. As in the earlier examples, we need to give maps on the second two skew product coordinates that link up the blocks in order to define T_1 and T_2. Before continuing, on $\bar{\Omega}$ introduce the partition

$$P = \{E \times [0,1)_1 \times [0,1)_2, F \times [0,1)_1 \times [0,1)_2, S \times [0,1)_1 \times [0,1)_2, S_0 \times [0,\tfrac{1}{2})_1 \times [0,1)_2, S_0 \times [\tfrac{1}{2},1)_1 \times [0,1)_2\} = \{\bar{E}, \bar{F}, \bar{S}, R, B\}.$$

It is in terms of this partition that all the arguments will be made. We will often speak of R and B as the colors red and black of a name.

Across 0-blocks, define

$$T_1(w,\theta,\phi) = (W(w),\theta,\phi) \quad \text{and} \quad T_2(w,\theta,\phi) = (W(w),\theta + \tfrac{1}{2},\phi).$$

Notice this looks exactly like the two maps with isomorphic squares. Suppose we know how T_1 and T_2 are defined across $(n-1)$-blocks. We now want to know how the second and third coordinates change from $(n-1)$-block to $(n-1)$-block in an n-block. Let $N_{i-1} \leqq n < N_i$. For this i, a point θ in $[0,1)_1$ can be written uniquely as

$$\theta = \sum_{j=1}^{i} \frac{n_j^i(\theta)}{p_j} + \theta' \quad \text{where} \quad 0 \leqq n_j^i(\theta) < p_j$$

$$\text{and} \quad \theta' < \frac{1}{p_1 \times \cdots \times p_i}.$$

We write (w,θ,ϕ) as $(w,\{n_1^i,\cdots,n_i^i,\theta'\},\phi)$, so that the second coordinate can be thought of as a set of $p_1 \times \cdots \times p_i$ points on which the group elements, given in the introduction, can act.

To think of ϕ as an element of G_1^i or G_2^i we do the following. Write

$$\phi = \sum_{j=1}^{\infty} \frac{\alpha_j}{2^j}, \quad \alpha_j = 0,1,$$

and define the sequences ϕ_i^1 and ϕ_i^2 inductively as follows: $\phi_1^1 = 0$, $\phi_1^2 = \pi$ and

$$\phi_{i+1}^1 = \begin{cases} \phi_i^1 & \text{if } \alpha_i = 0 \\ \phi_i^1 \pm \dfrac{1}{p_i} + \dfrac{1}{p_{i+1}} & \text{if } \alpha_i = 1 \end{cases} \qquad \text{and}$$

(2.3)

$$\phi_{i+1}^2 = \begin{cases} \phi_i^2 & \text{if } \alpha_i = 0 \\ \phi_i^2 \pm \dfrac{1}{p_i} + \dfrac{1}{p_{i+1}} & \text{if } \alpha_i = 1. \end{cases}$$

Remember, $\pm$ adds if $1/p_i$ is not a term in ϕ_i^1 and subtracts if it is. To save space we will write $\phi_i^{1(2)}$ to indicate a result holds for both indices. These inductive sequences $\phi_i^{1(2)}$ imbed in the ϕ coordinate the two subsets G_1^i and G_2^i as any $\phi_i^{1(2)} = (1/p_{i_1} + \cdots + 1/p_{i_k})$ where $i_1 < \cdots < i_k \leqq i$ and k is even (odd) and any such sequence occurs, and using Lebesgue measure on $[0,1)_2$, each such sequence is equally likely. We can write ϕ as $(\phi_i^{1(2)}, \alpha_i, \alpha_{i+1} \cdots)$.

With this representation for ϕ and θ, we can write the analogues of the maps in the introduction,

$$g_i^1(\theta,\phi) = (\theta + \phi_i^1, \phi) \quad \text{and} \quad g_i^2(\theta,\phi) = (\theta + \phi_i^2, \phi). \tag{2.4}$$

As was indicated earlier, we need a collection of maps on these two coordinates which move any two coordinates to nearly any other, if only to guarantee ergodicity of the skew products. We now define these. The first is rotation on θ, i.e.

$$f_\alpha(\theta,\phi) = (\theta + \alpha, \phi). \tag{2.5}$$

The map which moves ϕ is more subtle. The reason for this is that the isomorphism map on the p_kth power for some ϕ coordinates increases the rate of rotation of θ by $1/p_k$, and on some decreases it by $1/p_k$. To link together two such ϕ coordinates in a way that is consistent with this isomorphism we must switch the orientation of θ, as we pass through the map, on multiples of $1/p_k$. It will not work to simply switch the orientation of the entire θ coordinate, as this linking map must not only be consistent with the isomorphism at p_k, but all primes less than p_{i+1}. Thus on some p_k's the orientation must be preserved and on some, reversed, depending on how ϕ changes across the map.

Proceed as follows. Let $0 < l,\ k \leqq i$ and write θ as

$$\{n_1^i(\theta)\cdots n_l^i(\theta)\cdots n_k^i(\theta)\cdots n_i^i(\theta), \theta^i\} \text{ and } \phi \text{ as } \{\phi_i^{1(2)}, \alpha_i, \cdots\}.$$

Now

$$\begin{aligned} &f_{l,k}^i(\{n_1^i(\theta)\cdots n_l^i(\theta)\cdots n_k^i(\theta)\cdots n_i^i(\theta), \theta^i\}, \{\phi_i^{1(2)}, \alpha_i \cdots\}) \\ &\quad = \Big(\{n_1^i(\theta)\cdots - n_l^i(\theta)\cdots - n_k^i(\theta)\cdots n_i^i(\theta), \theta^i\}, \\ &\qquad \Big\{\phi_i^{1(2)} \pm \frac{1}{p_l} \pm \frac{1}{p_k}, \alpha_i \cdots\Big\}\Big). \end{aligned} \tag{2.6}$$

Thus $f_{l,k}^i$ switches the sign mod p_l and p_k of n_l^i and n_k^i and adds or deletes $1/p_l$ and $1/p_k$ from $\phi_i^{1(2)}$.

The maps $f_{l,k}^i$ generate a group under composition of order 2^{i-1} which is transitive on the $\phi_i^{1(2)}$ part of the coordinates. Let $f_1^i \cdots f_{2^i-1}^i$ be a sequence of elements from this group such that any element in the group is of the form $f_j^i \circ f_{j-1}^i \cdots \circ f^i$ for some j, and $f_{2^i-1} \circ f_{2^{i-1}-1} \cdots \circ f_1 = \text{identity}$.

Now we are ready to define the maps which link up the second and third coordinates in $(n-1)$-blocks in an n-block. We will call a map, which tells how the second and third coordinates change from the beginning of one $(n-1)$-block to the beginning of another in an n-block, a linking map.

There are $2^{i-1}(p_1 \times \cdots \times p_{i+1})^2 2^n$ $(n-1)$-blocks in an n-block. Let w_k be the first point in the kth $(n-1)$-block in the n-block. Let $W'(w_k) = w_{k+1}$. To define T_1 and T_2 across n-blocks, we need only to define $T_1^{l_k}$ and $T_2^{l_k}$ on (w_k, θ, ϕ). If $k \neq 0 \bmod (p_1 \times \cdots \times p_i)$, then

$$T_1^{l_k}(w_k, \theta, \phi) = (w_{k+1}, g_i^1(\theta, \phi)) \quad \text{and} \quad T_2^{l_k}(w_k, \theta, \phi) = (w_{k+1}, g_i^2(\theta, \phi)), \tag{2.7}$$

i.e. across segments of an n-block of lengths containing $(p_1 \times \cdots \times p_i)$ $(n-1)$-blocks the linking map between $(n-1)$-blocks is the analogue of the example in the introduction.

Let $Q_i = p_1 \times \cdots \times p_i$. If $k = k' \circ Q_i$, $k' \neq 0 \bmod Q_i$ then

$$\begin{aligned} &T_1^{l_k}(w_k, \theta, \phi) = (w_{k+1}, f_{Q_i^{-1}} \circ g_i^1(\theta, \phi)) \text{ and} \\ &T_2^{l_k}(w_k, \theta, \phi) = (w_{k+1}, f_{Q_i^{-1}} \circ g_i^2(\theta, \phi)), \end{aligned} \tag{2.8}$$

i.e. in between the strings of $(p_1 \times \cdots \times p_i)$ $(n-1)$-blocks on which the linking maps are already given, we also insert a rotation of the θ coordinate by Q_i^{-1}.

If $k = k' \circ Q_i^2$, then

$$\begin{aligned} &T_1^{l_k}(w_k, \theta, \phi) = (w_{k+1}, f^i_{(k' \bmod 2^i-1)} \circ f_{Q_i^{-1}} \circ g_i^1(\theta, \phi)) \text{ and} \\ &T_2^{l_k}(w_k, \theta, \phi) = (w_{k+1}, f^i_{(k' \bmod 2^i-1)} \circ f_{Q_i^{-1}} \circ g_i^2(\theta, \phi)) \end{aligned} \tag{2.9}$$

i.e. at these points we also add in an f_j^i, cycling through the j in order.

In order to get some feeling for this construction, consider the following. Let $w = (w, \theta, \phi)$ be an initial point in an n-block. Then w, $T_1(w) \cdots T_1^{h(n)}(w)$ is the trajectory of w through that n-block. Let $T^{l_0}(w)$, $T^{l_1}(w) \cdots T^{l_s}(w)$, $s = 2^{i-1}(Q_i)^2 2^n - 1$, be the initial points in the $(n-1)$-blocks in this trajectory. Write $T^{l_r}(w) = (w_r, \theta_r, \phi_r)$. Now $l_r \bmod Q_i$ cycles through the values $0 \cdots (Q_i - 1)$ from our choice for spacers. Writing

$$\theta_r = (n_1^i(\theta_r) \cdots n_i^i(\theta_r), \theta_r^i),$$

the various linking maps will cause $(n_1^i(\theta_r) \cdots n_i^i(\theta_r))$ to vary with r. Finally writing

$$\phi_r = (\phi_{r,i}^{1(2)}, \alpha_i \cdots),$$

the $\phi_{r,i}^{1(2)}$ will also vary with r. The following lemma gives some justification for the complexity of the linking maps and will be used later to show T_1 and T_2 are K-automorphisms.

LEMMA 1. *For any initial point w in an n-block, $N_{i-1} \leqq n < N_i$, the sequence of triples $(l_r \bmod Q_i;\ \{n_1^i(\theta_r) \cdots n_i^i(\theta_r)\};\ \phi_{r,i}^{1(2)})$ for $r = 0 \cdots 2^{i-1}(Q_i)^2 2^n - 1$ takes on each possible value $(p_{i+1}^2 \cdot 2^n)$ times.*

PROOF. We first show that the triple for $r' = r + k(2^{i-1}(Q_i)^2)$ is the same as that at r. This is true for $l_{r'}$ as we have $h(n-1) = 0 \bmod Q_i$, even if $n = N_{i-1} - 1$, and the initial F segment has length $= 0 \bmod Q_i$, and the S spacers between blocks have length $-1 \bmod Q_i$. Now for the other two terms, if we write out the linking maps between $k(2^{i-1}(Q_i)^2)$ and $(k+1)(2^{i-1}(Q_i)^2)$ as a product we get from (2.7), (2.8) and (2.9)

$$(g_i^{1(2)(Q_i)} \circ f_{Q_i^{-1}})^{Q_i} \circ f_{2^{i-1}}^i (g_i^{1(2)(Q_i)} \circ f_{Q_i^{-1}})^{Q_i} \circ f_{2^{i-1}-1}^i \cdots (g_i^{1(2)(Q_i)} \circ f_{Q_i^{-1}})^{Q_i} \circ f_1^i.$$

But as $g_i^{1(2)}$ has order dividing Q_i, and so does $f_{Q_i^{-1}}$, this is $f_1^i \circ f_2^i \cdots \circ f_{2^{i-1}}^i =$ identity. Thus the triples have a cycle of this length. It remains, then, to show that each value is taken exactly once in a cycle.

Across such a sequence of r's, first fix the value $l_r \bmod Q_i = 0$. Now the linking map between the first and kth block of such form is

$$f_{kQ_i^{-1}} \circ f^i \circ \cdots \circ f_2^i \circ f_1^i = \hat{f}_k \quad \text{where} \quad j = \left[\frac{k}{Q_i}\right]. \tag{2.10}$$

In this sequence, the maps $f_j^i \circ \cdots \circ f_1^i$ move $\phi_i^{1(2)}$ transitively, and $f_{kQ_i^{-1}}$ fixes $\phi_i^{1(2)}$ but moves $(n_1^i \cdots n_i^i)$ transitively. Hence the maps $\hat{f}_k$ are transitive on the last two terms of the triple.

Now if $l_r = t \bmod Q_i$ the linking map between the first and kth such block is $(g_i^{1(2)})^{-t}(\hat{f}_k)(g_1^{1(2)})^t$. As the $\hat{f}_k$ were transitive on the last two terms, and $g_i^{1(2)}$ is one to one onto on these terms, the above linking maps are also transitive, and the result follows.

We want now to add to this that the values $\{n_1^i(\theta) \cdots n_i^i(\theta)\}$ and $\phi_i^{1(2)}$ are determined by a points P-name across an N_i-block.

LEMMA 2. *Let $\bar{w} = (w, \theta, \phi)$ and $\bar{w}' = (w, \theta', \phi')$ be two initial points in an n-block, $N_{i-1} \leqq n < N_i$. The $T_{1(2)}$, $(\bar{E}, \bar{F}, \bar{S}, R, B)$, $h(n)$-names of $\bar{w}$ and $\bar{w}'$ are identical iff $(n_1^i(\theta) \cdots n_i^i(\theta)) = (n_1^i(\theta') \cdots n_i^i(\theta'))$ and $\phi_i^{1(2)} = \phi_i'^{1(2)}$.*

PROOF. If $(n_1^i(\theta) \cdots n_i^i(\theta)) = (n_1^i(\theta') \cdots n_i^i(\theta'))$ and $\phi_i^{1(2)} = \phi_i'^{1(2)}$, the names must be identical, as none of the linking maps up to this stage effect the values θ',

θ'' or $(\alpha_i \cdots)$, $(\alpha_i' \cdots)$, and all θ with a given fixed $(n_1^i(\theta) \cdots n_i^i(\theta))$ have the same color.

For the only if, let $\bar{w}_0, \bar{w}_1, \bar{w}_2 \cdots \bar{w}_s$ and $\bar{w}_0', \bar{w}_1', \bar{w}_2' \cdots \bar{w}_s'$ be, as in Lemma 1, the points on $\bar{w}$ and $\bar{w}$'s trajectories which lie at the beginning of the $(n-1)$-blocks in the n-block. Note that the linking map between the 0-th and the $k(Q_i)$th, $(N_i - 1)$-block, $k < Q_i$, is, from (2.10), $f_{kQ_i^{-1}}$, i.e. rotation of f by kQ_i^{-1}. If θ and θ' did not both lie in the same interval $[k'/Q_i, (k'+1)/Q_i)$ then one of these would have $\theta_{kQ_i} \in R$ and $\bar{\theta}'_{kQ_i} \in B$, or vice versa. Hence

$$(n_1^i(\theta) \cdots n_i^i(\theta)) = (n_1^i(\theta') \cdots n_i^i(\theta')).$$

Arguing similarly we know that the angles θ_1 and θ_1' for $\bar{w}$ and $\bar{w}'$ lie in the same interval $[k''/Q_i, (k''+1)/Q_i)$. The linking map between $\theta = \theta_0$ and θ_1 is g_i^1 which adds $\phi_i^{1(2)}$ to θ, and between $\theta' = \theta_0'$ and θ_1' adds $\theta_i'^{1(2)}$ to θ'. Hence $\phi_i^{1(2)} = \phi_i'^{1(2)}$ and we are done.

III. $T_1^{p_i}$ and $T_2^{p_i}$ are isomorphic

We want to show that there is a measurable measure preserving map H_{p_i} such that $T_2^{p_i} H_{p_i} = H_{p_i} T_1^{p_i}$. Furthermore, the construction of H_{p_i} is such that it can be expressed as a finite code from T_1, $(\bar{E}, \bar{F}, \bar{S}, R, B)$-names to T_2, $(\bar{E}, \bar{F}, \bar{S}, R, B)$-names.

The map H_{p_i} will be the identity map on uncolored names. On the (θ, ϕ) coordinates it will be defined across N_i-blocks as follows. Define

$$h_i^{1(2)}(\theta, \phi) = \left(\theta \mp \frac{1}{p_i}, \phi\right), \tag{3.1}$$

where by $\mp$ we mean, take the sign opposite to that in $\phi_{i+1}^{1(2)} \pm 1/p_i$, i.e. add $1/p_i$ to θ if $1/p_i$ is in $\phi_{i+1}^{1(2)}$, and subtract if not. Further, define the map

$$G_{p_i}^{1(2)}(\theta, \phi) = \left(\theta, \phi_{i+1}^{1(2)} \pm \frac{1}{p_i}, \alpha_{i+1} \cdots\right), \tag{3.2}$$

the analog of the isomorphism map in the introduction taking ϕ_{i+1}^1 to some $\bar{\phi}_{i+1}^2$.

Now for $0 \leqq k < h(N_i)$ and, as we said, $\bar{w}$ an initial point in an N_i-block, define

$$H_{p_i}(T_1^k(\bar{w})) = T_2^k(w, (h_i^1)^k G_{p_i}^1(\theta, \phi)). \tag{3.3}$$

We will later define H_{p_i} on the space not in N_i-blocks. This changes ϕ_i^1 by adding or deleting the $1/p_i$ term, and rotates θ, as we move down the name, by $\pm 1/p_i$ each step, depending on the nature of ϕ_i^1. From this it is clear that across

N_i-blocks at least H_{p_i} is an isomorphism on p_i-powers. We wish to verify this across all n-blocks, $n \geqq N_i$. We do this by showing (3.3) will always hold across such n-blocks.

In order to show this, we will need the following algebraic facts on the various maps on the (θ, ϕ) coordinates.

LEMMA 3. *Recall the following maps*;

$$f_\alpha(\theta, \phi) = (\theta + \alpha, \phi), \tag{3.4}$$

$$g_j^{1(2)}(\theta, \phi) = (\theta + \phi_j^{1(2)}, \phi), \tag{3.5}$$

$$\begin{aligned} f_{l,k}^j(\theta, \phi) &= f_{l,k}^j((n_1^j(\theta)\cdots n_l^j(\theta)\cdots n_k^j(\theta)\cdots n_j^j(\theta), \theta'), (\phi_j^{1(2)}, \alpha_j \cdots)) \\ &= \left((n_1^j(\theta)\cdots - n_l^j(\theta)\cdots - n_k^j(\theta)\cdots n_j^j(\theta), \theta')\left(\phi_j^{1(2)} \pm \frac{1}{p_l} \pm \frac{1}{p_k}, \alpha_j \cdots\right)\right), \end{aligned} \tag{3.6}$$

$$h_i^{1(2)}(\theta, \phi) = \left(\theta \mp \frac{1}{p_i}, \phi\right) \tag{3.7}$$

where $\pm$ has the opposite sign to $\phi_{i+1}^{1(2)} \pm 1/p_i$ (notice in this $\phi_{i+1}^{1(2)}$ could be replaced by an $\phi_j^{1(2)}$, $j > i$),

$$G_{p_i}^{1(2)}(\theta, \phi) = \left(\theta, \left(\phi_{i+1}^{1(2)} \pm \frac{1}{p_i}, \alpha_{i+1} \cdots\right)\right) \tag{3.8}$$

(notice here also that $i + 1$ could be replaced by any $j > i$).

On these maps the following facts hold, for $i < j$:

$$f_\alpha h_i^{1(2)} = h_i^{1(2)} f_\alpha, \tag{3.9}$$

$$f_\alpha G_{p_i}^{1(2)} = G_{p_i}^{1(2)} f_\alpha, \tag{3.10}$$

$$f_{l,k}^j G_{p_i}^{1(2)} = G_{p_i}^{1(2)} f_{l,k}^j, \tag{3.11}$$

$$f_{l,k}^j h_i^{1(2)} = h_i^{1(2)} f_{l,k}^j, \tag{3.12}$$

$$h_i^{1(2)} g_j^{1(2)} = g_j^{1(2)} h_i^{1(2)}, \tag{3.13}$$

$$h_i^{2(1)} G_{p_i}^{1(2)} g_j^{1(2)} = g_j^{2(1)} G_{p_i}^{1(2)}. \tag{3.14}$$

PROOF. Notice (3.9), (3.10) and (3.13) are clear. To prove (3.11), apply the fact that

$$\left(\phi_j^{1(2)} \pm \frac{1}{p_l}\right) \pm \frac{1}{p_k} = \left(\phi_j^{1(2)} \pm \frac{1}{p_k}\right) \pm \frac{1}{p_l}$$

at the appropriate places.

To prove (3.12), consider

$$f'_{l,k}h_i^{1(2)}(\theta,\phi)=f'_{l,k}h_i^{1(2)}(\theta,(\phi_j^{1(2)},\alpha_j\cdots))$$

$$=f'_{l,k}\left(\begin{Bmatrix}\theta-\dfrac{1}{p_i}, & \text{if } \dfrac{1}{p_i} \text{ is not in } \phi_j^{1(2)}\\ \theta+\dfrac{1}{p_i}, & \text{if } \dfrac{1}{p_i} \text{ is in } \phi_j^{1(2)}\end{Bmatrix},\quad (\phi_j^{1(2)},\alpha_j\cdots)\right)$$

$$=\left(\begin{Bmatrix}\theta'-\dfrac{1}{p_i} & \text{if } \dfrac{1}{p_i} \text{ is not in } \phi_j^{1(2)} \text{ and } i\neq l,k\\ \theta'+\dfrac{1}{p_i} & \text{if } \dfrac{1}{p_i} \text{ is in } \phi_j^{1(2)} \text{ and } i\neq l,k\\ \theta'+\dfrac{1}{p_i} & \text{if } \dfrac{1}{p_i} \text{ is not in } \phi_j^{1(2)} \text{ and } i=l,k\\ \theta'-\dfrac{1}{p_i} & \text{if } \dfrac{1}{p_i} \text{ is in } \phi_j^{1(2)} \text{ and } i=l,k\end{Bmatrix},\left(\phi_j^{1(2)}\pm\frac{1}{p_k}\pm\frac{1}{p},\alpha_j\cdots\right)\right)$$

where θ' is the first term in $f'_{l,k}(\theta,\phi)$,

$$=h_i^{1(2)}\left(\theta',\left(\phi_j^{1(2)}\pm\frac{1}{p_k}\pm\frac{1}{p_1},\alpha_j\cdots\right)\right.$$

$$=h_i^{1(2)}f'_{l,k}(\theta,\phi).$$

It is this fact which is critical in the form of $f'_{l,k}$.

To prove (3.14) consider

$$h_i^{2(1)}G_{p_i}^{1(2)}g_j^{1(2)}(\theta,\phi)=h_i^{2(1)}G_{p_i}^{1(2)}g_j^{1(2)}(\theta,(\phi_j^{1(2)},\alpha_j\cdots))$$

$$=h_i^{2(1)}G_{p_i}^{1(2)}(\theta+\phi_j^{1(2)},(\phi_j^{1(2)},\alpha_j\cdots))=h_i^{2(1)}\left(\theta+\phi_j^{1(2)},\left(\phi_j^{1(2)}\pm\frac{1}{p_i},\alpha_j\cdots\right)\right)$$

$$=\left((\theta+\phi_j^{1(2)})\,\textcircled{\pm}\,\frac{1}{p_i},\left(\phi_j^{1(2)}\pm\frac{1}{p_i},\alpha_j\cdots\right)\right),$$

where $\textcircled{\pm}$ has the same sign as $\pm$ in the second term,

$$=\left(\theta+\left(\phi_j^{1(2)}\pm\frac{1}{p_i}\right),\left(\phi_j^{1(2)}\pm\frac{1}{p_i},\alpha_j\cdots\right)\right)=g_j^{2(1)}\left(\theta,\left(\phi_j^{1(2)}\pm\frac{1}{p_i},\alpha_j\cdots\right)\right)$$

$$=g_j^{2(1)}G_{p_i}^{1(2)}(\theta,\phi).$$

This identity is really just a reinterpretation of the isomorphism map in the introduction.

THEOREM 1. *For a.e.* $\bar{w} \in \bar{\Omega}$, $H_{p_i}(T_1^{p_i}(\bar{w})) = T_2^{p_i} H_{p_i}(\bar{w})$.

PROOF. What we first verify is that $\bar{w}$ is the initial point in an n-block, $n \geqq N_i$, and $k < h(n)$, then

$$H_{p_i}(T_1^k(\bar{w})) = T_2^k((w, (h_i^1)^k \circ (G_{p_i}^1)(\theta, \phi)). \tag{3.13}$$

We already know this for N_i-blocks (3.3). Assume it true for n-blocks. Let $\bar{w}$ be an initial point in an $(n+1)$-block.

Case 1. Suppose $T_1^k(\bar{w})$ is the initial point in the lth n-block in this $(n+1)$-block, and $N_{j-1} \leqq n+1 < N_j$, $j > i$. Then

$$T_2^{-k} G_{p_i}(T^k(\bar{w})) = (w, (\bar{f}_l^2)^{-1} G_{p_i}^1 \bar{f}_l^1(\theta, \phi)) \tag{3.14}$$

where $\bar{f}_l^1$ and $\bar{f}_l^2$ are the linking maps for T_1 and T_2 between the 1st and lth n-blocks in an $(n+1)$-block. To write out these linking maps as a product, there is a set L of pairs (l, k), $l, k \leqq j$, and integers l_1 and l_2 with

$$\bar{f}_l^1 = (g_j^1)^{l_1} \circ f_{l_2 Q_j^{-1}} \circ \prod_{(l,k) \in L} f_{l,k}^l \quad \text{and} \tag{3.15}$$

$$\bar{f}_l^2 = (g_j^2)^{l_1} \circ f_{l_2 Q_j^{-1}} \circ \prod_{(l,k) \in L} f_{l,k}^l. \tag{3.16}$$

Thus,

$$\begin{aligned}
&(\bar{f}_l^2)^{-1} \circ G_{p_i}^1 \circ \bar{f}_l^1 \\
&= \Big(\prod_{(l,k)\in L} f_{l,k}^l\Big)^{-1} \circ f_{-l_2 Q_j^{-1}} \circ (g_j^2)^{-l_1} \circ G_{p_i}^1 (g_j^1)^{l_1} \circ f_{l_2 Q_j^{-1}} \circ \prod_{(l,k)\in L} f_{l,k}^l \\
&= \Big(\prod_{(l,k)\in L} f_{l,k}^l\Big)^{-1} \circ f_{l_2 Q_j^{-1}} \circ h_i^{-l_1} \circ G_{p_i}^1 \circ f_{l_2 Q_j^{-1}} \circ \prod_{(l,k)\in L} f_{l,k}^l, \text{ by (3.14)} \\
&= h_i^{-l_1} \circ G_{p_i}^1 \text{ by (3.10) and (3.11).}
\end{aligned}$$

Thus

$$T_2^{-k} H_{p_i}(T_1^k(\bar{w})) = (w, h_i^{-l_1} G_{p_i}^1(\theta, \phi)).$$

Now $k \bmod p_i = (-l) \bmod p_i$ as $h(n) = 0 \bmod p_i$. Further, $l_1 \bmod p_i = l \bmod p_i$ as both count the number, mod p_i, of the $g_j^{1(2)}$'s occurring in the linking map $\bar{f}^{1(2)}$. Hence $h_i^{-l_1} = h_i^k$ and we have the result in this case.

Case 2. Suppose $T_1^k(\bar{w})$ lies inside the lth n-block. Let $T_1^{k'}(\bar{w})$ be the initial point in the n-block containing $T_1^k(\bar{w})$. Using the induction statement on $(W^{k'}(w), \tilde{f}_l^1(\theta,\phi))$ and $k-k'$ we get

$$T_2^{-k}H_{p_l}T_1^k(\bar{w}) = (w, (\tilde{f}_l^2)^{-1}(h_l^1)^{k-k'}G_{p_l}^1\tilde{f}_l^1(\theta,\phi))$$
$$= (w, (h_l^1)^{k-k'}(\tilde{f}_l^2)^{-1}G_{p_l}^1\tilde{f}_l^1(\theta,\phi)),$$

as by Lemma 3, h_l^1 commutes with all linking maps,

$$= (w, (h_l^1)^{k-k'}(h_l^1)^kG_{p_l}^1(\theta,\phi))$$

by precisely the computation in Case 1,

$$= (w, (h_l^1)^kG_{p_l}^1(\theta,\phi))$$

and (3.14) again holds.

For those k with $T^k(\bar{w})$ not in an n-block, define $H_{p_l}(T^k(\bar{w}))$ by (3.14). Thus (3.14) always holds on $(n+1)$-blocks. By induction, H_{p_l} is defined everywhere and on all n-blocks, $n \geqq N_l$, satisfies (3.14).

Now to see that (3.14) implies isomorphism on p_lth powers, we know that for a.e. $\bar{w}$, there is an n with $\bar{w}$ and $T_1^{p_l}(\bar{w})$ both in the same n-block.

Let $\bar{w} = T_1^{k'}(\bar{w})$ where $\bar{w}'$ is an initial point in this n-block. Using (3.14),

$$T_2^{-p_l}(H_{p_l}T_1^{p_l}(\bar{w})) = T_2^{-p_l}(H_{p_l}T_1^{k'+p_l}(\bar{w}') = T_2^{k'}(\bar{w}', (h_l^1)^{k'+p_l}G_{p_l}^1(\theta',\phi')$$
$$= T_2^{k'}(\bar{w}', (h_l^1)^{k'}G_{p_l}^1(\theta',\phi') = H_{p_l}(T_1^{k'}(\bar{w}')) = H_{p_l}(\bar{w}).$$

THEOREM 2. *The map H_{p_l} and $(H_{p_l})^{-1}$ are finite codes on T_1, $(\bar{E},\bar{F},\bar{S},R,B)$-names to and from T_2, $(\bar{E},\bar{F},\bar{S},R,B)$-names.*

PROOF. Outside N_l-blocks, as H_{p_l} fixes uncolored names, H_{p_l} is surely a finite code on names as all colored points are in N_l-blocks. On N_l-blocks,

$$H_{p_l}(T_1^k(\bar{w})) = T_2^k(w, (h_l^1)^kG_{p_l}^1(\theta,\phi))$$

for $\bar{w}$ an initial point and $0 \leqq k \leqq h(N_l)$. If $\theta = (n_1^{l+1}(\theta)\cdots n_{l+1}^{l+1}(\theta), \theta^{l+1})$ and $\phi = (\phi_{l+1}^1, \alpha_{l+1}\cdots)$, then the map $(h_l^1)^kG_{p_l}(\theta,\phi)$ is determined by only the values $n_1^{l+1}(\theta)\cdots n_{l+1}^{l+1}(\theta)$ and θ_{l+1}^1 and fixes θ^{l+1} and $(\alpha_{l+1}\cdots)$. By Lemma 2, then $(h_l^1)^kG_{p_l}(\theta,\phi)$ is determined by the $T_1(\bar{E},\bar{F},\bar{S},R,B)$-name of w across the N_l-block, and determine only the $T_2(\bar{E},\bar{F},\bar{S},R,B)$-name of $H_{p_l}(w)$ across its N_l-block. The result follows.

IV. T_1 and T_2 are K-automorphisms

Using Lemmas 1 and 2, we will apply the standard argument on such transformations to show T_1 and T_2 are K-automorphisms. As the argument varies only slightly from that in [4], the reader is referred to this proof for details we skip here.

THEOREM 3. *Let $P = (\bar{E}, \bar{F}, \bar{S}, R, B)$. Given $\varepsilon > 0$ and integer $k > 0$, there is an N such that for any $m > 0$ and $n \geqq N$,*

$$\bigvee_{-m}^{0} T_1^i(P) \perp^{\varepsilon} \bigvee_{n-k}^{n+k} T_1^i(P). \tag{4.1}$$

PROOF. First fix $\varepsilon > 0$ and $k > 0$. Choose N_i large enough that the set of w in an N_i-block more than k-positions from the ends of the block is at least $(1 - \varepsilon^2)$. Call this set $\bar{\Omega}$. Let $N = h(N_{i+1} - 1) + 1$.

Now fix $n > N$, and let $\bar{\Omega}^n = T^{-n}(\bar{\Omega})$. As $\mu(\bar{\Omega}) > 1 - \varepsilon^2$, it will suffice to show

$$\bigvee_{-m}^{0} T_1^i(P)/\bar{\Omega}^n \perp \bigvee_{n-k}^{n+k} T_1^i(P)/\bar{\Omega}^n. \tag{4.2}$$

Let A be an atom of $\bigvee_{-m}^{0} T_1^i(P)/\bar{\Omega}^n$, i.e. a set with a fixed P-name from $-m$ to 0. We want the distribution of T_1, P-names on $T_1^{n-k}(A) \cdots T_1^{n+k}(A)$ to be exactly the same as that on $T_1^{n-k}(\bar{\Omega}^n) \cdots T_1^{n+k}(\bar{\Omega}^n)$. This last is precisely the distribution of $2k$-P-names on N_i-blocks.

On A define the functions $L(w)$, $F(w)$ and $N(w)$ as usual (see [4] and Fig. 1).

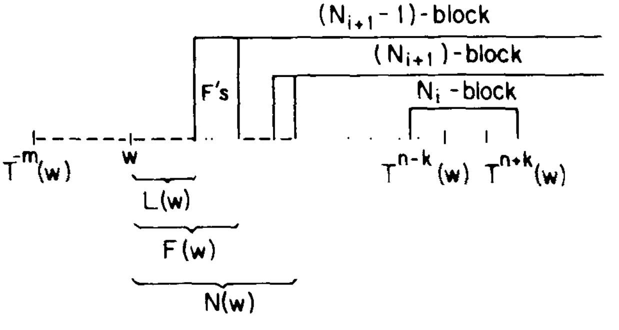

Fig. 1

Break A into subsets A_i with a fixed P-name on $T_1^{-m}(w)$ to $T_1^{L(w)}(w)$ and $T_1^{F(w)}(w) \cdots T_1^{N(w)}(w)$. In A_i, the block of F's from $T_1^{L(w)}(w)$ to $T_1^{F(w)}(w)$ can take any value $Q_{i+2} \cdots (N_{i+1} - 1)Q_{i+2}$, except those which do not put $T_1^{n-k}(w) \cdots T_1^{n+k}(w)$ into an N_i-block in the right $(N_i + 1)$-block. But each of those possible is equally likely. As $(N_{i+1}-1)Q_{i+2} > h(N_i + 1)$, this will put $T_1^{n-k}(w)\cdots T_1^{n+k}(w)$ into all the N_i-blocks in this $(N_i + 1)$-block. The sequence of positions $T(w)$ that can hit across this $(N_i + 1)$-block come in multiples of Q_{i+2}. Lemmas 1 and 2 tell us that in this $(N_i + 1)$-block, we see any possible P–N_i-block name at any distance mod Q_{i+2} into the $(N + 1)$-block, each with equal probability. Arguing as in Theorem 1.2 of [4], (4.2) follows, and hence the result.

V. T_1 and T_2 are nonisomorphic

In showing this a number of the lemmas depend on the block structure of the uncolored doubly infinite name of a point. These facts are shown in precisely the same manner as in previous papers [4], [5]. When this is true the proof will be omitted and the reader referred to such places.

As usual in such arguments, if w is a point of Ω, we can write the P, $T_{1(2)}$ name of w as $\cdots B_{-2}, B_{-1}, B_0, B_1, B_2 \cdots$ where $B_i \in P$ and $T_{1(2)}^i(w) \in B_i$. From our construction, such a name consists of colored n-blocks. Such an n-block is a sequence $B_i \cdots B_{h(n)+i-1}$. If T_1 and T_2 were isomorphic, then T_2, P-names could be approximated arbitrarily well in the $\bar{d}$ metric by finite codes of T_1, P-names. We will show that the block structure and colorings of T_2, P-names are so rigid under closeness in the $\bar{d}$ metric that this is impossible.

Lemma 4. *There is an $\bar{\varepsilon} > 0$ such that if $A_k \cdots A_{h(n)+k-1}$ and $B_j \cdots B_{h(n)+j-1}$ are any two W, (E, F, S_0, S)-names across n-blocks, and*

$$\sum_{i=0}^{n_1} f(i) < |k-j| < h(n)\left(1 - \frac{1}{2^{(n/2)}}\right),$$

then $A_i \neq B_i$ for at least $\bar{\varepsilon}(h(n) - |k-j|)$ values $\sup(j,k) \leqq i \leqq h(n) + \inf(j,k)$.

Proof. Argue exactly as in Lemma 1.1 of [4].

Now in an n-block $A = A_k \cdots A_{h(n)+k-1}$, let $(l, S)_A$ represent the Sth l-block in A, for $l \leqq n$.

Lemma 5. *If $A = A_k \cdots A_{k+h(n)-1}$ and $B = B_j \cdots B_{j+h(n)-1}$ are two n-block, W, (E, F, S_0, S)-names, $|k-j| < h(n)(1 - 1/2^{(n/2)})$, and $A_i = B_i$ for at least $(1 - \varepsilon\bar{\varepsilon})(h(n) - |k-j|)$ values $i \in \{\sup(j,k) \cdots h(n) + \inf(j,k)\}$, then all but at most 2ε of the $(0,S)_A$ blocks lie precisely over $(0,S)_B$.*

Proof. Argue exactly as in Lemma 1.2 of [4].

Now suppose there is an isomorphism map, $\Phi T_1 = T_2 \Phi$. As Φ is measurable, $\Phi^{-1}(P) \subset \bigvee_{-\infty}^{\infty} T_1^i(P)$. Thus for any ε there is an $N(\varepsilon)$ with $\Phi^{-1}(P) \subset^{\varepsilon} \bigvee_{-N(\varepsilon)}^{N(\varepsilon)} T_1^i(P)$.

That is to say, there is a $\bar{P}(\varepsilon) \subset \bigvee_{-N(\varepsilon)}^{N(\varepsilon)} T_1^i(P)$, and $|\bar{P}(\varepsilon) - \Phi^{-1}(P)| < \varepsilon$. Hence, for almost all $w \in \bar{\Omega}$, the P-name of $T_1^{-N+k}(w) \cdots T_1^{N+k}(w)$ will determine what atom of $\bar{P}(\varepsilon)$, $T_1^k(w)$ is in, and this $\bar{P}(\varepsilon)$-name agrees with the P-name of $T_2^k(\Phi(w))$ for all but a set of k's of density ε.

Let w_0 be a point which codes well for a sequence of ε's tending to 0, and a generic point for the T_1, P process. As usual, [4], let A, B, C denote occurrences of an n-block P-name in the P-name of w_0, i.e. a sequence $A_k(w) \cdots A_{k+h(n)-1}(w)$ which is the P-name across an n-block and $T^i(w) \in A_i(w)$. Further, let $\langle A \rangle_n$ or $\langle A - B \rangle_n$ or $\langle A - B - C \rangle_n$ denote the collection of all occurrences of the n-block P-name of A in the name of w_0, or the pair $A - B$, of the triple $A - B - C$, where any size spacers are allowed between A and B and C, but they are consecutive n-blocks in an $(n+1)$-block. An occurrence $A = A_k \cdots A_{k+h(n)-1}$ of an n-block in the T_1, P-name of w_0, $\bar{P}(\varepsilon)$-codes α-well if the $\bar{P}(\varepsilon)$ name across $T_1^k(w) \cdots T_1^{k+h(n)-1}(w)$ and the P-name across $T_2^k(\Phi(w)) \cdots T_2^{k+h(n)-1}(\Phi(w))$ differ in at most $\alpha h(n)$ places.

Lemma 6. *For all but $\sqrt[3]{2\varepsilon}$ of the classes $\langle A \rangle_n$ of n-block T_1, P-names, for all $A \in \langle A \rangle_n$ but a set of density $\sqrt[3]{2\varepsilon}$ in $\langle A \rangle_n$, A $\bar{P}(\varepsilon)$-codes $\sqrt[3]{2\varepsilon}$ well. Similarly for the classes $\langle A - B \rangle_n$ and $\langle A - B - C \rangle_n$.*

Proof. See Lemma 1.4 of [4].

Lemma 7. *Given $\varepsilon > 0$ there is an N so that if $n \geqq N$ then for all but at most ε of the classes $\langle A \rangle_n$, for all but a set of $A \in \langle A \rangle_n$ of density ε, $\Phi(A)$ contains $(1-\varepsilon)h(n)$ of a T_2, P n-block.*

Proof. See Lemma 1.6 of [4].

Lemma 8. *If A and $A' = T_1^k(A) \in \langle A \rangle_n$ are n-block names in the T_1, P-name of w, both of which $\bar{P}(\varepsilon)$-code $\frac{1}{4}$ $\bar{\varepsilon}$-well and $\Phi(A)$ and $\Phi(A')$ overlap T_2, P-n-blocks C and C' respectively, each in more than $\frac{3}{4}h(n)$ places, then $T_1^k(C)$ and C' overlap in at least $h(n) - \Sigma_{i=0}^{n} f(i)$ places, and across the segment of their length, where they overlap A' they differ in at most an $\bar{\varepsilon}$ fraction of the name.*

Proof. $T_2^k(C)$ and C' overlap in at least $\frac{1}{2}h(n)$ places, with at most an $\bar{\varepsilon}$ fraction of differences in this overlap. The result now follows from Lemma 4.

Lemma 9. *Given $\varepsilon > 0$ there is an N so that if $n \geqq N$ then for all but at most ε of the classes $\langle A \rangle_n$, for all but a set of $A \in \langle A \rangle_n$ of density ε, the n-block $\Phi(A)$ overlaps in more than $\frac{3}{4}h(n)$ places occupies the same position in an $(n+1)$-block as A.*

Proof. This is argued with only a minor variation as Lemma 3 of [3]. We can combine Lemmas 5, 6, 7, 8 and 9 into the following result.

Lemma 10. *If Φ is an isomorphism between T_1 and T_2, and $\varepsilon > 0$, then there is an N so that if $n \geqq N$, for all but ε of the T_1 n-block classes $\langle A \rangle_n$, for all but ε of the occurrences A, $A' = T^k(A) \in \langle A \rangle_n$, $\phi(A)$ overlaps a T_2–n-block C so that all the $(n-1)$-blocks in A overlap the corresponding $(n-1)$-blocks in C in at least $(1-\varepsilon)h(n-1)$ places. Similarly $\Phi(A')$ overlaps C' and $T^k(C)$ and C' have all but ε of their corresponding 0-blocks lying precisely over each other and of the same color.*

As in [4], we will now show that if blocks code as the above says they must, then the colors of the names will force a contradiction. The initial point of a j-block has three coordinates, the second two, (θ, ϕ), determine precisely the sequence of colors, red and black, of the 0-blocks down the name independent of the first coordinate. Define the metric $\|(\theta, \phi), (\theta', \phi')\|_j$ as the fraction of differences in the two 0-block color sequence along a j-block given by (θ, ϕ) and (θ', ϕ').

As an average over j-blocks we can write

$$\|(\theta, \phi), (\theta', \phi')\|_{j+1} = \frac{1}{2^{t-1}Q_t^2} \sum_{l,k,k'} \left\| \left(\prod_{l'=1}^{l} f_{l'}^{t-1} \right)(\theta + kQ_t^{-1} + k'\phi_t^2, \phi), \right. \tag{5.1}$$
$$\left. \left(\prod_{l'=1}^{l} f_{l'}^{t-1} \right)(\theta' + kQ_t^{-1} + k'\phi_t'^2, \phi') \right\|_j$$

where $N_{t-1} \leqq j+1 < N_t$. Thus this value is constant across j in the ranges $N_{t-1} \leqq j < N_t$. Noticing that the maps $f_{l'}^{t}(\theta, \phi)$ act independently on the two coordinates, we write

$$\left(\prod_{l'=1}^{l} f_l^{t-1} \right)(\theta, \phi) = ({}_1f_l^{t-1}(\theta) + \theta', {}_2f_l^{t-1}(\phi)). \tag{5.2}$$

Lemma 11. *The value $\|(\theta, \phi), (\theta', \phi)\|_j$ is independent of ϕ.*

Proof. True for $j = 1$ as ϕ has no effect on 1-block names. For $j + 1$, using the above comments,

$$\begin{aligned} \|(\theta, \phi), (\theta', \phi)\|_{j+1} &= \frac{1}{2^{t-1}Q_t^2} \sum_{l,k,k'} \|({}_1f_l^{t-1}(\theta + kQ_t^{-1} + k'\phi_t^2){}_2f_l^{t-1}(\phi)), \\ &\qquad ({}_1f_l^{t-1}(\theta' + kQ_t^{-1} + k\phi_t^2){}_2f_l^{t-1}(\phi))\|_j \\ &= \frac{1}{2^{t-1}Q_t} \sum_{l,k} \|({}_1f_l^{t-1}(\theta) + kQ_t^{-1}, {}_2f_l^{t-1}(\phi)), \\ &\qquad ({}_1f_l^{t-1}(\theta') + kQ_t^{-1}, {}_2f_l^{t-1}(\phi))\|_j \end{aligned} \tag{5.3}$$

as ${}_1f_i^{i-1}(\theta + qQ_i^{-1}) = {}_1f_i^{i-1}(\theta) + q'Q_i^{-1}$ where the map q to q' is linear and invertible and $(kQ_i^{-1} + k'\phi_i^1)$ takes on each value qQ_i^{-1} exactly Q_i times. By induction, this sum is independent of ϕ.

LEMMA 12. *For any θ, θ', ϕ and prime p_n,*

$$\sum_{k=1}^{p_n} \left\| \left(\theta + \frac{k}{p_n}, \phi\right), (\theta', \phi) \right\|_j \geqq \left[\frac{p_n}{3}\right]\frac{1}{6} \quad \text{for all } j \ (=1 \text{ if } p_n = 2). \tag{5.4}$$

PROOF. For $j < N_1$ it is easy to see that $\sum_{k=1}^{p_n} \|(\theta + k/p_n, \phi), (\theta', \phi)\|_j \geqq [p_n/2]$. Assume the result for $j \leqq J$. The result for $J+1$ breaks into two cases.

Case 1. If $n \leqq i$ and $N_{i-1} \leqq J+1 < N_i$, then

$$\sum_{k=1}^{p_n} \left\| \left(\theta + \frac{k}{p_n}, \phi\right), (\theta', \phi) \right\|_{J+1} = \frac{1}{2^{i-1}Q_i} \sum_k \sum_{l,k'} \left\| \left({}_1f_l^{i-1}\left(\theta + \frac{k}{p_n}\right) + k'Q_i^{-1}, {}_2f_l^{i-1}(\phi)\right), ({}_1f_l^{i-1}(\theta') + k'Q_i^{-1}, {}_2f_l^{i-1}(\phi)) \right\|_J$$

as in Lemma 11. But this

$$= \frac{1}{2^{i-1}Q_i} \sum_{l,k'} \left(\sum_k \left\| \left({}_1f_l^{i-1}(\theta) + k'Q_i^{-1} + \frac{k}{p_n}, {}_2f_l^{i-1}(\phi)\right), ({}_1f_l^{i-1}(\theta') + k'Q_i^{-1}, {}_2f_l^{i-1}(\phi)) \right\|_J \right)$$

by the form of ${}_1f_l^{i-1}$. Thus whatever bound holds for J will hold for $J+1$. (This completes the result for $p_n = 2$.)

Case 2. If $n > i$, $N_{i-1} \leqq J+1 < N_i$, then

$$\sum_{k=1}^{p_n} \left\| \left(\theta + \frac{k}{p_n}, \phi\right), (\theta', \phi) \right\|_{J+1} = \sum_k \left(\frac{1}{2^{i-1}} \sum_{l,k'} \left\| \left({}_1f_l^{i-1}\left(\theta + \frac{k}{p_n}\right) + k'(2Q_i)^{-1}, \phi\right), ({}_1f_l^{i-1}(\theta') + k'Q_i^{-1}, \phi) \right\|_J \right. \tag{5.5}$$

by Lemma 11.

The map ${}_1f_l^{i-1}$ on any $\theta = (n_1^{i-1} \cdots n_i^{i-1}, \theta')^{-1}$ switches the sign on some even number of n_k^{i-1}'s. Let ${}_1f_{l^*}^{i-1}$ be that one which switches the signs precisely where ${}_1f_l^{i-1}$ does not, except for n_2^{i-1} which it switches or fixes as ${}_1f_l^{i-1}$ does, and n_1^{i-1} which it switches or fixes in order to switch altogether an even number. Now we can write the sum (5.5) as

$$\sum_k \left(\frac{1}{2^l Q_l}\sum_{l,k'}\left[\left\|\left({}_1f_l^{l-1}\left(\theta+\frac{k}{p_n}\right)+k'Q_l^{-1},\phi\right),\left({}_1f_l^{l-1}(\theta')+k'Q_l^{-1},\phi\right)\right\|_J\right.\right.$$

$$\left.+\left\|\left({}_1f_l^{l-1}\left(\theta+\frac{k}{p_n}\right)+k'Q_l^{-1},\phi\right),\left({}_1f_l^{l-1}(\theta')+k'Q_l^{-1},\phi\right)\right\|_J\right]$$

$$(5.6)\qquad =\sum_k\left(\frac{1}{2^l Q_l}\sum_{l,k}\left[\left\|\left({}_1f_l^{l-1}\left(\theta+\frac{k}{p_n}\right)-{}_1f_l^{l-1}(\theta')+k'Q_l^{-1},\phi\right),(k'Q_l^{-1},\phi)\right\|_J\right.\right.$$

$$\left.+\left\|\left({}_1f_l^{l-1}\left(\theta+\frac{k2\pi}{p_n}\right)-{}_1f_l^{l-1}(\theta')+k'Q_l^{-1},\phi\right),(k'Q_l^{-1},\phi)\right\|_J\right]$$

by rearranging the sums.

Now by the triangle inequality and simplifications, (5.6) is

$$(5.7)\qquad \geqq \frac{1}{2^l Q_l}\sum_{l,k,k'}\left\|\left(\frac{2(n_2^{k,l}-n_2'^l)}{3}+k'Q_l^{-1},\phi\right),(k'Q_l^{-1},\phi)\right\|_J$$

where $n_2^{k,l}$ is the $\frac{1}{3}$ term in ${}_1f_l^{l-1}(\theta+k/p_n)$ and $n_2'^l$ is the $\frac{1}{3}$ term in ${}_1f_l^{l-1}(\theta')$. It is easy to show, as $p_n > p_l$, that for fixed l, $n_l^{k,l}$ is -1, 0 and 1 each at least $[p_l/3]$ times. Hence grouping (5.7) over constant $(n_2^{k,l}-n_2'^l)$ we get that it is

$$(5.8)\qquad \begin{aligned}&\geqq \frac{1}{2Q_l}\left[\frac{p_l}{3}\right]\sum_{k=1}^{3}\sum_{k'}\left\|\left(k'Q_l^{-1}+\frac{k}{3},\phi\right)(k'Q_l^{-1},\phi)\right\|_J\\ &\geqq \frac{1}{2}\left[\frac{p_l}{3}\right]\inf_\theta\sum_{k=1}^{3}\left\|\left(\theta+\frac{k}{3},\phi\right)(\theta,\phi)\right\|_{N_2}\end{aligned}$$

by Case 1. It is easy to compute a bound of $\frac{1}{3}$ for this infimum and we are done.

LEMMA 13. *If $\phi_i^2 \neq \phi_i'^2$, then for $j \geqq N_i$,*

$$\|(\theta,\phi),(\theta',\phi')\|_j \geqq \tfrac{1}{30}.$$

PROOF. We proceed by induction on i. As $\phi_1^1=\phi_1'^1$ always, there is nothing to say here. Assume the result for j where $j+1=N_i$. Then

$$\|(\theta,\phi),(\theta',\phi')\|_{j+1}$$

(5.9)

$$=\frac{1}{2^l Q_{l+1}^2}\sum_{l,k,k'}\|({}_1f_l^i(\theta+kQ_i^{-1}+k'\theta_i^1),{}_2f_l^i(\phi)),({}_1f_l^i(\theta'+kQ_i^{-1}+k'\phi_i^{-1}),{}_2f_l^i(\phi')\|_{N_i-1}.$$

If $\phi_{i-1}^1\neq\phi_{i-1}'^1$ we are done by the induction hypothesis on each term. If $\phi_{i-1}^1=\phi_{i-1}'^1$ and $\phi_i^1\neq\phi_i'^1$, then $\phi_i^1=\phi_i'^1\pm 1/p_i$. Assume w.l.o.g. that the sign is $+$. Then (5.9) becomes

$$\frac{1}{2^l Q_{i+1}^2}\sum_{l,k,k'}\left\|\left({}_1f_i^l\left(\theta+kQ_i^{-1}+k'\phi_i'^1+\frac{k'}{p_i}\right),{}_2f_i^l(\phi)\right),\right.$$
$$\left.{}_1f_i^l(\theta'+kQ_i^{-1}+k'\phi_i'^1),{}_2f_i^l(\phi')\right\|_{N_{i-1}}. \tag{5.10}$$

As $\|\cdot,\cdot\|_{N_{i-1}}$ depends only on $\phi_{i-1}^1=\phi_{i-1}'^1$, Lemma 10 applies to each term in (5.10), and it is

$$\geqq\left[\frac{p_i}{3}\right]\frac{1}{6p_i}\geqq\frac{1}{30},\quad\text{as}\quad p_i\geqq 3.$$

We are now ready to combine Lemma 10 and Lemmas 12 and 13 to force further restrictions on Φ to the point of contradiction.

LEMMA 14. *Let Φ be an isomorphism of T_1 and T_2. Let $A\in\langle A\rangle_n$ have initial coordinates $(T_1^k(w),\theta,\phi)$, and the block C which $\phi(A)$ overlaps have initial coordinates $(T_2^{k'}(\Phi(w)),\bar\theta,\bar\phi)$. Given $\varepsilon>0$, there is an N_i so that if $n\geqq N_i$ for all but ε of the classes $\langle A\rangle_n$, for all but ε of the occurrences $A\in\langle A\rangle_n$, $\bar\phi_i^2$ depends only on ϕ_i^1.*

PROOF. If ε is chosen $<1/30$ in Lemma 10, then by going from $\langle A\rangle_n$ to $\langle A'\rangle_n$ where A and A' have the same (θ,ϕ) coordinates, by first changing the second half of the name, then the first, Lemma 13 implies the result.

LEMMA 15. *Let Φ be an isomorphism of T_1 and T_2. Let $A\in\langle A\rangle_n$ have initial coordinates $(T_1^k(w),\theta,\phi)$ and the block C which $\Phi(A)$ overlaps have initial coordinates $(T_2^{k'}(\Phi(w)),\bar\theta,\bar\phi)$. Given $\varepsilon>0$, there is an N_i so that if $n\geqq N_i$, for all but ε of the classes $\langle A\rangle_n$, for all but ε of the occurrences $A\in\langle A\rangle_n$, any prime p_j not in ϕ_i^1 is also not in $\bar\phi_i^2$.*

PROOF. Fix p_j and let $N_{i+1}\geqq n>N_i\geqq N_{j-1}$. If $n<N_{j-1}$ neither ϕ_i^1 nor $\bar\phi_i^2$ contain p_j.

Consider the set of all classes $\langle A\rangle_n$ where p_j is not in the ϕ_i^1 coordinate of A. This is half the classes. For such an $\langle A\rangle_n$, let $P=Q_i/p_j$, and break the $(n-1)$-blocks in A into strings of length Q_i. In each such the (θ,ϕ) coordinates of the $(n-1)$-blocks are periodic with period P. Hence we can write this collection of $\langle A\rangle_n$ as a collection of sequences

$$\langle A_0\rangle_n\cdots\langle A_{p_j-1}\rangle_n \tag{5.11}$$

where $\langle A_{r+1}\rangle_n$ is obtained from $\langle A_r\rangle_n$ by rotating the $(n-1)$-blocks in a string of $Q_i(n-1)$-blocks through the period P in the first half of the block, fixing those in the second half. Call this permutation of $(n-1)$-block names R.

Let δ be some positive number we will specify later. By Lemma 14, if n is large enough independent of p_l we know that after deleting δ of the terms $\langle A_r \rangle_n$ in the sequence, except for δ of the $A_r \in \langle A_r \rangle_n$, $\Phi(A_r)$ overlaps an n-block C_r whose $\bar{\phi}$ coordinate is the same for all these C_r, and all but δ of the $(n-1)$-blocks in A_r overlap the corresponding $(n-1)$-blocks in C_r in $(1-\delta)$ $h(n-1)$ places. And if we write out an array of names

$$(5.12)\qquad \begin{array}{c} \langle C_0 \rangle_n \\ R^{-1}(\langle C_1 \rangle_n) \\ \cdot \\ \cdot \\ \cdot \\ R^{-p_l+1}(\langle C_{p_l-1} \rangle_n) \end{array}$$

in this array the terms differ by at most $\bar{\varepsilon}\delta^2/2$ places, except for δ of the names in the array (remember $\bar{\varepsilon}$ is the constant in Lemma 4). Hence, by Lemma 5, for all but δ of the C_i, for all but δ of the $(n-1)$-blocks in $R^{-r}(\langle C_r \rangle_n)$ and $R^{-s}(\langle C_s \rangle_n)$, all but δ of the 0-blocks in corresponding $(n-1)$-blocks have the same color. That is, the color sequences across these good $(n-1)$-blocks differ pairwise by at most δ.

Now suppose p_l *is* in $\bar{\phi}_1^2$. Let the lth $(n-1)$-block in C_r have $(\bar{\theta}, \bar{\phi})$ coordinate $(\bar{\theta}_{r,l}, \bar{\phi}_l)$ (I am suppressing the fact that δ of the C_r do not have $\bar{\phi}$ as third coordinate).

(5.13) Looking at an l in the first half of an n-block we get from above that in the $(n-1)$-block color sequences for coordinates

$$(\bar{\theta}_{0,l}, \bar{\phi}),\ \left(\bar{\theta}_{1,l} + \frac{k}{p_l}, \bar{\phi}_l\right) \cdots \left(\bar{\theta}_{p_l-1,l} + \frac{k(p_l-1)}{p_l}, \bar{\phi}_l\right)$$

except for δ choices of l and δ terms in the string, differ pairwise by less than δ, where $k = P \bmod p_l$.

(5.14) Looking at l' in the second half of an n-block we get that in the $(n-1)$-block color sequences for coordinates

$$(\bar{\theta}_{0,l'}, \bar{\phi}_{l'}),\ (\bar{\theta}_{1,l'}, \bar{\phi}_{l'}) \cdots (\bar{\theta}_{p_l-1,l'}, \bar{\phi}_{l'}),$$

except for δ choices of l' and δ terms in the string, differ pairwise by less than δ.

(5.15) For l' halfway along the n-block from l, $(\bar{\theta}_{s,l}, \bar{\phi}_l)$ and $(\bar{\theta}_{s,l'}, \bar{\phi}_{l'})$ have identical color sequences.

Combining (5.13), (5.14) and (5.15) we can conclude that deleting $\sqrt{6\delta}$ choices of l and $\sqrt{6\delta}$ terms in the sequence, the color sequences for

$$(\bar{\theta}_{r,l}, \bar{\phi}_l), \left(\bar{\theta}_{r,l} + \frac{k}{p_l}, \bar{\phi}_l\right) \cdots \left(\bar{\theta}_{r,l} + \frac{k(p_l - 1)}{p_l}, \bar{\phi}_l\right)$$

differ pairwise by less than 2δ.

But for such a good l we conclude that there is a k' so that

$$\sum_{k=0}^{p_l-1} \left\| \left(\bar{\theta}_{r,l} + \frac{k'}{p_l}, \bar{\phi}_l\right), \left(\bar{\theta}_{r,l} + \frac{k}{p_l}, \bar{\phi}_l\right) \right\| \leqq p_l(2\delta + \sqrt{6\delta}).$$

But for δ chosen small enough this conflicts with Lemma 12. The result now follows from Lemma 14.

THEOREM 4. *T_1 and T_2 are nonisomorphic.*

PROOF. Assume otherwise. By Lemma 15, for n large enough for all but ε of the n-blocks A, $\Phi(A)$ overlaps a block C and any prime p_l not in the ϕ coordinate of A is also not in the $\bar{\phi}$ coordinate of C. Interchange the roles of T_1 and T_2 in this and the ϕ^1_n coordinate of A must be identical to the $\bar{\phi}^2_n$ coordinate of C for n large enough. This is impossible.

VI. Conclusion

This construction, although quite intricate in places, is basically very simple. The intricacy really only arises because of the particular nature of the example we were after. The basic idea is quite straightforward. The rigidity in $\bar{d}$ of the block structure of the original Ornstein example can be used to carry an algebraic system into the class of K-automorphisms. Beyond this, it is just a matter of finding the proper fiber structure for a skew product so that the usual arguments can proceed.

What is perhaps most significant in this example is that all the isomorphisms are given as finite codes on a single partition. In fact, if this were not so the example would be even more difficult to discuss, as the explicitness of these maps would be lost. As such, though, it gives some indication of how delicate the names of a K-automorphism can be made.

There remains still one major question in this direction. Can two nonisomorphic K-flows be constructed which are isomorphic at all times? The difficulty here is the need for an uncountable collection of isomorphisms which must, of necessity, get close to each other vaguely.

It would also be valuable to try to put these constructions into a general framework, so that arguments which, as we have seen, are nearly identical for the different examples, need be given only once in suitable generality.

REFERENCES

1. J. Clark, *A Kolmogorov shift with no roots.*

2. D. S. Ornstein, *A K-automorphism with no square root and Pinsker's conjecture*, Advances in Math. **10** (1973), 63–68.

3. D. S. Ornstein and P. Shields, *An uncountable family of K-automorphisms*, Advances in Math. **10** (1973), 89–102.

4. D. Rudolph, *Two nonisomorphic K-automorphisms with isomorphic squares*, Israel J. Math. **23** (1976), 274–287.

5. D. Rudolph, *Two K-flows isomorphic at a dense set of times but not all times*, Ph.D. Dissertation, Stanford University, 1975.

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